NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 1

Assessment Task 2 Term 1 2012

Name:_______________________________ **Mathematics Class:**_______

Time Allowed: 55 minutes + 2 minutes reading time

Available Marks: 38

Instructions:

Question 1 (a) Multiple choice (5 marks)

- Indicate your answer by colouring the appropriate circle on the answer sheet provided

Question 1 b) and c), Question 2 and Question 3 Free response

- Write your answers on the examination booklet provided
- Write on one side of the page only
- Do not work in columns
- Attempt all questions
- Show all necessary working
- Marks may be deducted for incomplete or poorly arranged work

Question	1 a	1 bc	2ab	2c	3a	3 bc	Total	
PE3	/2				/6		/8	
PE4		/7					/7	
HE2				/4			/4	
HE7	/3	/3	/7			/6	/19	
							/38	

Question 1 (15 marks)

(a) **Multiple Choice** (1 mark each) Answer on the multiple choice answer sheet provided

(i) The largest four digit number to be found in the arithmetic sequence 2, 9, 16, 23, … is

(A) 9995 (B) 9996 (C) 9998 (D) 9999

(ii) The series $2^x + 5\times 2^{-x} + 25\times 2^{-3x} + \ldots\ldots\ldots$ can be expressed as

(A) $\sum_{k=1}^{\infty} 5^{1-k}2^{-2x}$ (B) $\sum_{k=1}^{\infty} 5^{k-1}2^{(3-2k)x}$ (C) $\sum_{k=1}^{\infty} 5^{k-1}2^{3-2k}$ (D) $\sum_{k=0}^{\infty} 5^{k}2^{x}$

(iii) Observe that $1^3+2^3+3^3=(1+2+3)^2$ and $1^3+2^3+3^3+4^3=(1+2+3+4)^2$.
If the same pattern works for all positive integers, then which of the following expressions is equivalent to $\sqrt{1^3+2^3+3^3+\ldots.+k^3}$?

(A) $\frac{k}{2}(k+1)$ (B) $\left[\frac{k}{2}(k+1)\right]^2$ (C) $\sqrt{(1+2+3+\ldots..+k)}$ (D) $\sqrt{k^3}$

(iv) The parametric equations $x=2t^2$ and $y=3-t$ have the Cartesian equation

(A) $y=3-\frac{x}{2}$ (B) $x=2(3-y)^2$ (C) $x=\frac{(3-y)^2}{2}$ (D) $x^2=2(3-y)^2$

(v) Given $(p+q)^2=8pq$ and T has coordinates $(a(p+q),apq)$ then the locus of T is

(A) $x^2=8ay$ (B) $x^2=\frac{8y}{a}$ (C) $y=\frac{8x^2}{a}$ (D) $y=8ax^2$

Question 1 continued on next page

Question 1 (continued) Use a new booklet

(b) On the way to work, Kelly passes through a particular intersection. She notices that the traffic lights are red for 2 minutes, amber for 20 seconds and green for 1 minute.

(i) Find the probability that the lights will be green as a fraction in simplest terms. **1**

(ii) Find the probability that at least one light will be green if Kelly drives through the intersection on three successive occasions. **2**

(c) (i) Show that the equation of the tangent to $x^2 = 12y$ at the point $(6t, 3t^2)$ is $y = tx - 3t^2$. **3**

(ii) Find the values of t for which the tangent could pass through the point $(13, -10)$. **2**

(iii) Hence state the equations of the tangents to $x^2 = 12y$ passing through $(13, -10)$. **2**

Question 2 (11 marks) Use a new booklet

(a) Find the value(s) of k for which the series $3 + 3k^2 + 3k^4 + \dots\dots\dots$ has a limiting sum of $\frac{49}{8}$. **3**

(b) In a TV game show, Bella is to spin the prize wheel twice. The wheel is divided into sectors labelled WIN or LOSE. If the wheel stops on a WIN both times she wins a major prize, if it stops on a WIN only once she wins a minor prize.

Let the probability that the wheel stops on WIN be *p,* where $p < 0.5$.

(i) What is the probability that the wheel will stop on LOSE, in terms of p? **1**

(ii) Given that the probability that Bella wins a minor prize is 32%, find the value of p. **2**

(iii) Find the probability that she wins a major prize, as a fraction in simplest terms. **1**

(c) Prove by Mathematical Induction that **4**

$$\frac{1}{2\times5} + \frac{1}{5\times8} + \frac{1}{8\times11} + \dots\dots\dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)}$$

Question 3 (12 marks) Use a new booklet

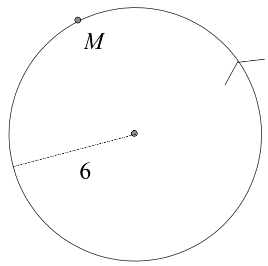
(a) The distinct points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$. The points P and Q are constrained to move such that $QR \perp PR$ where R is the fixed point $\left(2a, a\right)$ which also lies on the parabola.

(i) Show that $pq + p + q = -5$ **2**

(ii) If Q has coordinates $\left(-4a, 4a\right)$ find the coordinates of P. **2**

(iii) By drawing a diagram, or otherwise, find the values of p and q for which the relationship $QR \perp PR$ is not possible. **2**

(b) An electric train travels along a circular track of radius 6 metres. At an instant selected at random, the current is cut off and the train stopped. What is the probability that it stops with less than 2 metres of track between the train and the station M ? Give your answer as an exact value. **2**



(c) In a certain arithmetic series, the ratio of the sum of 10 terms to the sum of 5 terms is $13:4$. The sum of the first 20 terms is 115.

(i) Show that $\dfrac{2a+9d}{a+2d} = \dfrac{13}{4}$ **2**

(ii) Find the first term and the common difference. **2**

End of paper

Q1 Solutions 2012 Assessment March

a) $2 + (n-1)7 < 10000$

$n-1 < 1428.286\ldots$

$\therefore n-1 = 1428$

$T_n = 2 + 1428 \times 7$

$= 9998$ (C)

(ii) Index for 5 must be 5^{k-1}

Indices for 2 go down $x, -x, -3x$ etc

which is arithmetic: $T_n = x + (k-1)(-2x)$

$= 3x - 2kx$

$= (3-2k)x$

$\therefore$ Index for 2: $2^{(3-2k)x}$ (B)

(iii) Clearly $1^3 + 2^3 + 3^3 + \ldots + k^3 = (1+2+3+\ldots+k)^2$

$= S_k$ of series $a=1, d=1$

$= \left[\frac{k}{2}(k+1)\right]^2$

Remembering to square root answer:

$\sqrt{1^3+2^3+3^3+\ldots+k^3} = \frac{k}{2}(k+1)$ (A)

(iv) $x = 2t^2$ and $y = 3-t$

Sub $t = 3-y$ into $x = 2t^2$

$x = 2(3-y)^2$ (B)

(v) $p+q = \frac{x}{a}$ and $pq = \frac{y}{a}$ $\Rightarrow \left(\frac{x}{a}\right)^2 = \frac{8y}{a}$ (A)

$\therefore x^2 = 8ay$

Question 1 (cont)

b) (i) $P(\text{green}) = \dfrac{60}{120+20+60}$

$= \dfrac{3}{10}$

(ii) $P(\text{at least one green}) = 1 - P(\text{no green at all})$

$= 1 - \left(\dfrac{7}{10}\right)^3$

$= \dfrac{657}{1000}$

c) $y = \dfrac{1}{12}x^2$

$\dfrac{dy}{dx} = \dfrac{x}{6}$

At $(6t, 3t^2)$, $\dfrac{dy}{dx} = \dfrac{6t}{6}$

$= t$

Equation of tangent is $y - 3t^2 = t(x - 6t)$

$y = tx - 3t^2$

(ii) $(13, -10)$ will satisfy tangent equation

$-10 = 13t - 3t^2$

$3t^2 - 13t - 10 = 0$

$(3t+2)(t-5) = 0$

$\therefore t = -2/3$ or $t = 5$

(iii) When $t = -2/3$, equation of tangent is $y = \dfrac{-2x}{3} - 3\left(\dfrac{-2}{3}\right)^2$

$y = \dfrac{-2x}{3} - \dfrac{4}{3}$

or

$2x + 3y + 4 = 0$

When $t = 5$, equation of tangent is

$y = 5x - 75$

Question 2

(a) $3 + 3k^2 + 3k^4 + \dots$

$$S_\infty = \frac{a}{1-r}$$

$$\frac{49}{8} = \frac{3}{1-k^2}$$

$$49 - 49k^2 = 24$$

$$0 = 49k^2 - 25$$

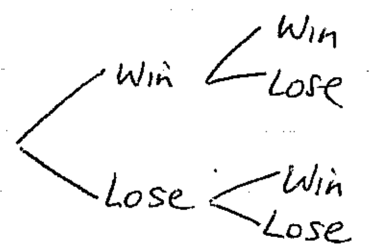
$$= (7k-5)(7k+5)$$

$k = \pm \frac{5}{7}$ both solutions are valid

(b) $p < 0.5$

(i) $P(\text{LOSE}) = 1 - p$

(ii)



WW major prize

WL – minor prize

LW – minor prize

$$P(WL) + P(LW) = 0.32$$

$$p(1-p) + (1-p)p = \frac{8}{25}$$

$$2p - 2p^2 = \frac{8}{25}$$

$$0 = 25p^2 - 25p + 4$$

$$= (5p-1)(5p-4)$$

$p = \frac{1}{5}$ or $p = 4/5$

But $p < 0.5$

$\therefore p = \frac{1}{5}$

(iii) $P(WW) = P(\text{major prize}) = \left(\frac{1}{5}\right)^2 = \frac{1}{25}$

Question 2 (cont)

c) $$\frac{1}{2\times5}+\frac{1}{5\times8}+\frac{1}{8\times11}+\cdots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)}$$

Test $n=1$

$$LHS=\frac{1}{10}$$

$$RHS=\frac{1}{2(5)}$$

$$\therefore LHS = RHS$$

Assume there is a value $n=k$ for which statement holds

ie assume $\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{2(3k+2)}$

We wish to show true for $n=k+1$

ie show that $\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3k+2)(3k+5)}=\frac{k+1}{2(3k+5)}$

$$LHS=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)} \quad \text{by assumption}$$

$$=\frac{k(3k+5)+2}{2(3k+2)(3k+5)}$$

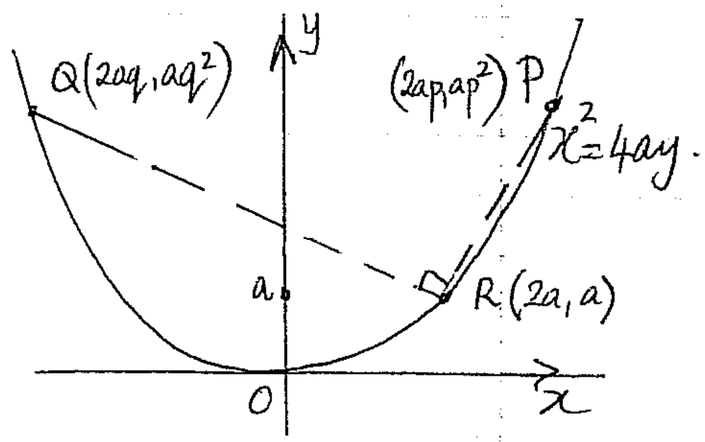
$$=\frac{3k^2+5k+2}{2(3k+2)(3k+5)}$$

$$=\frac{(3k+2)(k+1)}{2(3k+2)(3k+5)}$$

$$=\frac{k+1}{2(3k+5)}$$

$\therefore$ By mathematical Induction the result holds for all $n\geq1$

Question 3



<u>Note</u> $R(2a, a)$ must lie on right hand side because $a > 0$, by definition

$QR \perp PR$

$\therefore m_{QR} \times m_{PR} = -1$

$$\frac{aq^2 - a}{2aq - 2a} \times \frac{ap^2 - a}{2ap - 2a} = -1$$

$$\frac{a(q+1)(q-1)}{2a(q-1)} \times \frac{a(p-1)(p+1)}{2a(p-1)} = -1$$

$$\frac{q+1}{2} \times \frac{p+1}{2} = -1$$

$$pq + p + q + 1 = -4$$

$$\therefore pq + p + q = -5 \quad *$$

(ii) If Q is $(-4a, 4a)$ then $q = -2$ since $2aq = -4a$

$$-2p + p - 2 = -5 \quad \text{from } *$$
$$p = 3$$

$\therefore P$ is $(6a, 9a)$

(iii) A right angle at R cannot be formed if

- P or Q lie on R itself
- P or Q lie on the other end of the latus rectum, ie at $(-2a, a)$

$\therefore p \neq \pm 1$ or $q \neq \pm 1$

<u>Note</u> if $p = -1$, equation $*$ becomes $-q - 1 + q = -5$ which is impossib

Question 3 (cont)

b) Train can stop <u>either side</u> of the station ; Circumference $= 12\pi$

$\therefore P(\text{stops within } 2\text{ m of } M) = \dfrac{4}{2 \times 6 \times \pi}$

$= \dfrac{1}{3\pi}$

c) $S_{10} = \frac{10}{2}(2a + 9d)$

$= 5(2a + 9d)$

$S_5 = \frac{5}{2}(2a + 4d)$

$= 5(a + d)$

$\therefore \dfrac{13}{4} = \dfrac{5(2a+9d)}{5(a+2d)}$

$\dfrac{13}{4} = \dfrac{2a+9d}{a+2d}$

(ii) $S_{20} = \frac{20}{2}(2a + 19d)$

$115 = 20a + 190d$

$23 = 4a + 38d \quad (1)$

From (i) $4(2a+9d) = 13(a+2d)$

$0 = 5a - 10d \quad (2)$

(1) $\times 5$ $\quad 115 = 20a + 190d$

(2) $\times 4$ $\quad 0 = 20a - 40d$

$115 = 230d$

$\therefore d = 1/2$

$\therefore a = 1$

$\therefore$ first term is 1 and common difference is $\frac{1}{2}$